ENGINEERING MECHANICS - STATICS
CHAPTER-2

[FORCE SYSTEMS]

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Force:

In the study of statics, we are concerned with two fundamental quantities: length or distance, which requires no explanation, and force. The quantity length can be seen with the eye but with force, the only thing that is ever seen is its effect. We can see a spring being stretched or a rubber ball being squashed but what is seen is only the effect of a force being applied and not the force itself. With a rigid body there is no distortion due to the force and in statics it does not move either. Hence, there is no visual indication of forces being applied. Force cannot be seen or measured directly but must always be imagined. Generally, the existence of some force requires little imagination but to imagine all the different forces which exist in a given situation may not be too easy. Furthermore, in order to perform any analysis, the forces must be defined precisely in mathematical terms. For the moment we shall content ourselves with a qualitative definition of force. A force is that quantity which tries to move the object on which it acts. This qualitative definition will suffice for statical problems in which the object does not move but we shall have to give it further consideration when we study the subject of dynamics. If the object does not move, the force must be opposed and balanced by another force. So, we identify the force as:

A "force" is an action that changes, or tends to change, the state of motion of the body upon which it acts. It is a vector quantity that can be represented either mathematically or graphically

A complete description of a force MUST include its:

1. MAGNITUDE
2. DIRECTION and SENSE
3. POINT OF ACTION

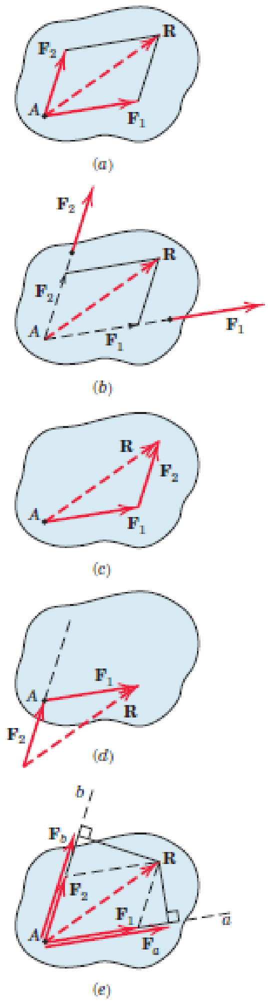


Figure 2/3

Forces are classified as either ***contact*** or ***body*** forces. And may be further classified as either ***concentrated*** or ***distributed*** Forces. Every contact force is actually applied over a finite area and is therefore really a distributed force. However, when the dimensions of the area are very small compared with the other dimensions of the body, we may consider the force to be concentrated at a point with negligible loss of accuracy. Force can be distributed over an *area*, as in the case of mechanical contact, over a *volume* when a body force such as weight is acting, or over a *line*, as in the case of the weight of a suspended cable. The ***weight*** of a body is the force of gravitational attraction distributed over its volume and may be taken as a concentrated force acting through the center of gravity.

Resultant (R): If the Two forces **F**1 and **F**2 or more forces are *concurrent at a point* and their lines of action intersect at that a common point of application, or the two concurrent forces lie in the same plane but are applied at two different points, so, by the principle of transmissibility, Thus, they can be added using the parallelogram law in their common plane to obtain their sum.

We can also use the triangle law to obtain **R**, but we need to move the line of action of one of the forces.

If we add the same two forces as shown in Fig. 2/3*d*, we correctly preserve the magnitude and direction of **R**, but we lose the correct line of action, because **R** obtained in this way does not pass-through *A*. Therefore, this type of combination should be avoided.

We can express the sum of the two forces mathematically by the vector equation

$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$

Finding the Components of a Force.

The relationship between a force and its vector components along given axes must not be confused with the relationship between a force and its **perpendicular projections** (also called ***orthogonal*** projections.) onto the same axes. Figure 2/3*e* shows the perpendicular projections $\mathbf{F}_a$ and $\mathbf{F}_b$ of the given force **R** onto axes *a* and *b*, which are parallel to the vector components $\mathbf{F}_1$ and $\mathbf{F}_2$ of Fig. 2/3*a*. Figure 2/3*e* shows that the components of a vector are not necessarily equal to the projections of the vector onto the same axes. Furthermore, the vector sum of the projections $\mathbf{F}_a$ and $\mathbf{F}_b$ is not the vector **R**, because the parallelogram law of vector addition must be used to form the sum. The components and projections of **R** are equal only when the axes *a* and *b* are perpendicular.

RECTANGULAR COMPONENTS

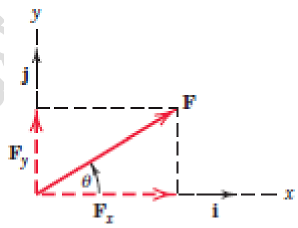


Figure 2/5

The most common two-dimensional resolution of a force vector is into rectangular components. It follows from the parallelogram rule that the vector **F** of Fig. 2/5 may be written as

$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$ **(2/1)**

Where **F***x* and **F***y* are ***vector components*** of **F** in the *x*- and *y*-directions.

$\mathbf{F} = F_x\mathbf{i} + F_y\,\mathbf{j}$ **(2/2)**

$Fx = F\cos\theta \qquad F = \sqrt{Fx^2 + Fy^2}$

$Fy = F\sin\theta \qquad \theta = \tan^{-1}\frac{Fy}{Fx}$

$\mathbf{R} = \mathbf{F}1 + \mathbf{F}2 = (F_{1x}\,\mathbf{i} + F_{1y}\,\mathbf{j}) + (F_{2x}\,\mathbf{i} + F_{2y}\,\mathbf{j})$

Or

$R_x\mathbf{i} + R_y\,\mathbf{j} = (F_{1x} + F_{2x})\mathbf{i} + (F_{1y} + F_{2y})\,\mathbf{j}$

$Rx = F_{1x} + F_{2x} = \sum F_x$

$Ry = F_{1y} + F_{2y} = \sum F_y$

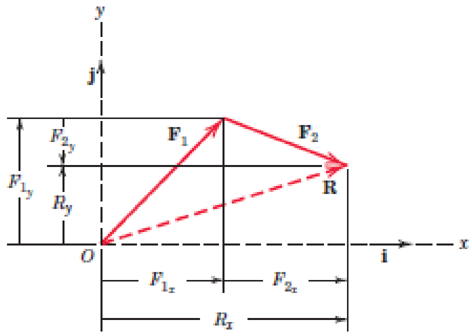


Figure 2/7

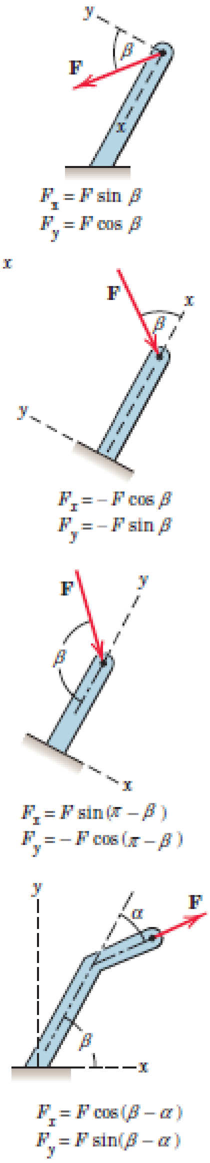
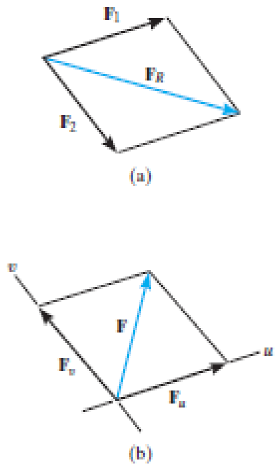


Figure 2/6

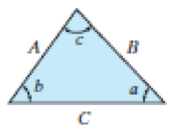
Parallelogram Law.

- Two "component" forces F_1 and F_2 in Fig. 2–10 a. add according to the parallelogram law, yielding a *resultant* force F_R that forms the diagonal of the parallelogram.
- If a force F is to be resolved into *components* along two axes *u* and *v*, Fig. 2–10 b, then start at the head of force F and construct lines parallel to the axes, thereby forming the parallelogram. The sides of the parallelogram represent the components, F_u and F_v.
- Label all the known and unknown force magnitudes and the angles on the sketch and identify the two unknowns as the magnitude and direction of F_R, or the magnitudes of its components.



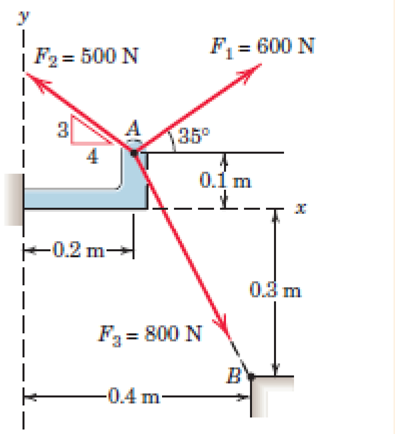
Trigonometry.

- Redraw a half portion of the parallelogram to illustrate the triangular head-to-tail addition of the components.
- From this triangle, the magnitude of the resultant force can be determined using the law of cosines, and its direction is determined from the law of sines. The magnitudes of two force components are determined from the law of sines. The formulas are given in Fig. 2–10 c.



(C.W.) Sample Problem 2/1

The forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, all of which act on point *A* of the bracket, are specified in three different ways. Determine the *x* and *y* scalar components of each of the three forces.



Solution. The scalar components of $\mathbf{F}_1$, from Fig. *a*, are

$F_{1x} = 600 \cos 35^o = 491$ N *Ans.*

$F_{1y} = 600 \sin 35^o = 344$ N *Ans.*

The scalar components of **F**2, from Fig. *b*, are

$F_{2x} = -500(\frac{4}{5}) = -400$ N *Ans.*

$F_{2y} = 500\,(\frac{3}{5}) = 300$ N *Ans.*

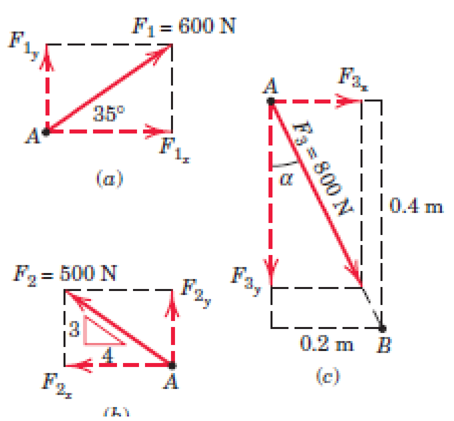
Note that the angle which orients $\mathbf{F}_2$ to the *x*-axis is never calculated. The cosine and sine of the angle are available by inspection of the 3-4-5 triangle. Also note that the *x* scalar component of $\mathbf{F}_2$ is negative by inspection.

The scalar components of $\mathbf{F}_3$ can be obtained by first computing the angle α of Fig. *c*.

$\alpha = tan^{-1}\,[\frac{0.2}{0.4}] = 26.6^o$ *Ans.*

Then $F3x = F3 \sin \alpha = 800 \sin 26.6^o = 358$ N *Ans.*

$F_{3y} = -F_3 \cos \alpha = -800 \cos 26.6^o = -716$ N



Alternatively, the scalar components of $\mathbf{F}_3$ can be obtained by writing $\mathbf{F}_3$ as a magnitude times a unit vector $\mathbf{n}_{AB}$ in the direction of the line segment *AB*. Thus,

$$\mathbf{F}_3 = F_3\mathbf{n}_{AB} = F_3\,\frac{\overrightarrow{AB}}{\overline{AB}} = 800\,\left[\frac{(0.2)\mathbf{i} - (0.4)\mathbf{j}}{\sqrt{(0.2)^2 + ((-0.4)^2}}\right]$$

$$= 800\,[0.447\mathbf{i} - 0.894\mathbf{j}]$$

$$= 358\mathbf{i} - 716\mathbf{j} \text{ N}$$

The required scalar components are then

$F_{3x} = 358$ N *Ans.*

$F_{3y} = -716$ N *Ans.*

which agree with our previous results.

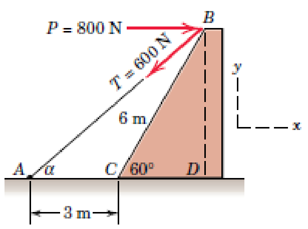
Helpful Hints

1- You should carefully examine the geometry of each component determination problem and not rely on the blind use of such formulas as $F_x = F \cos \theta$ and $F_y = F \sin \theta$.

2- A unit vector can be formed by dividing *any* vector, such as the geometric position vector $\overrightarrow{AB}$, by its length or magnitude. Here we use the over arrow to denote the vector which runs from *A* to *B* and the overbar to determine the distance between *A* and *B*.

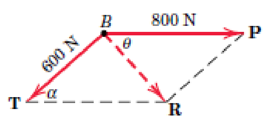
Sample Problem 2/2

Combine the two forces **P** and **T**, which act on the fixed structure at *B*, into a single equivalent force **R**.



Graphical solution. The parallelogram for the vector addition of forces **T** and **P** is constructed as shown in Fig. *a*. The scale used here is 1 cm = 400 N; a scale of 1 cm = 100 N would be more suitable for regular-size paper and would give greater accuracy. Note that the angle *a* must be determined prior to construction of the parallelogram. From the given figure

$$tan \propto = \frac{\overrightarrow{BD}}{\overrightarrow{AD}} = \frac{6\,sin60}{3+6\cos 60} = 0.866 \qquad \propto = 40.9°$$



(a)

Measurement of the length *R* and direction θ of the resultant force **R** yields the approximate results

$R = 525$ N $\theta = 49^o$ *Ans.*

Geometric solution. The triangle for the vector addition of **T** and **P** is shown in Fig. *b*. The angle _ is calculated as above. The law of cosines gives

$R^2 = (600)^2 + (800)^2 - 2(600)(800) \cos 40.9° = 274{,}300$

$R = 524$ N *Ans.*

From the law of sines, we may determine the angle θ which orients **R**. Thus,

$$\frac{600}{\sin\theta} = \frac{534}{\sin 40.9°} \qquad \sin\theta = 0.750 \qquad \theta = 48.6° \ Ans.$$

Algebraic solution. By using the *x-y* coordinate system on the given figure, we may write

$Rx = \Sigma F_x = 800 - 600 \cos 40.9°$ $\quad\theta = 346$ N *Ans.*

The magnitude and direction of the resultant force **R** as shown in Fig. *c* are then

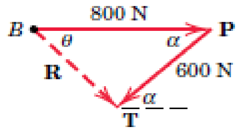
$$R = \sqrt{Rx^2 + Ry^2} = \sqrt{(346)^2 + (-393)^2} = 524 \ N \ Ans.$$

$$\theta = \tan^{-1}\frac{|Ry|}{|Rx|} = \tan^{-1}\frac{393}{346} = 48.6°$$

The resultant **R** may also be written in vector notation as

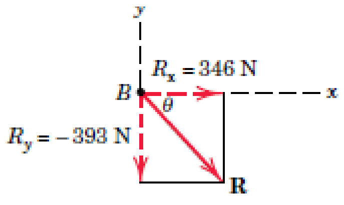
$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} = 346\mathbf{i} - 393\mathbf{j}$ N *Ans.*

1- Note the repositioning of **P** to permit parallelogram addition at *B*.



(*b*)

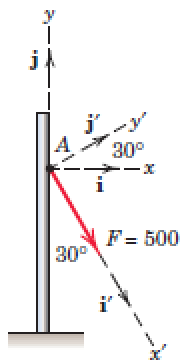
2-Note the repositioning of **F** so as to preserve the correct line of action of the resultant **R**.



(*c*)

(C.W.) Sample Problem 2/3

The 500-N force **F** is applied to the vertical pole as shown. (1) Write **F** in terms of the unit vectors **i** and **j** and identify both its vector and scalar components. (2) Determine the scalar components of the force vector **F** along the *x*' and *y*' axes. (3) Determine the scalar components of **F** along the *x* and *y*'-axes.



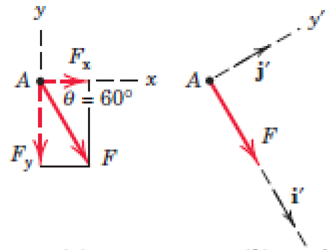
Solution. ***Part (1).*** From Fig. *a* we may write **F** as

$$\mathbf{F} = (F\cos\theta)\,\mathbf{i} - (F\sin\theta)\,\mathbf{j}$$
$$= (500\cos 60°)\,\mathbf{i} - (500\ son\ 60°)\mathbf{j}$$
$$= (250\,\mathbf{i} - 433\,\mathbf{j})\mathbf{N} \qquad Ans.$$

The scalar components are $Fx = 250$ N and $Fy = -433$ N. The vector components are $\mathbf{F}_x = 250\mathbf{i}$ N and $\mathbf{F}_y = -433\mathbf{j}$ N.

Part (2). From Fig. *b* we may write **F** as **F** = 500**i** N, so that the required scalar components are

$F_{x'} = 500\ N \quad F_{y'} = 0$ *Ans.*

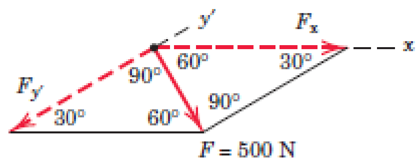


(*a*) (*b*)

Part (3). The components of **F** in the *x*- and *y* -directions are nonrectangular and are obtained by completing the parallelogram as shown in Fig. *c*. The magnitudes of the components may be calculated by the law of sines. Thus,

$$\frac{|Fx|}{\sin 90°} = \frac{500}{\sin 30°} \qquad |Fx| = 1000\ N$$

$$\frac{|Fy'|}{\sin 60°} = \frac{500}{\sin 30°} \qquad |Fy'| = 866\ N$$



(*c*)

The required scalar components are then

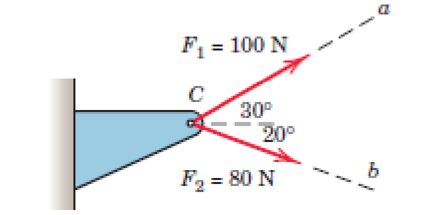
$F_x = 1000$ N $F_{y'} = -866$ N *Ans.*

Helpful Hint

1- Obtain F_x and F_y_ graphically and compare your results with the calculated

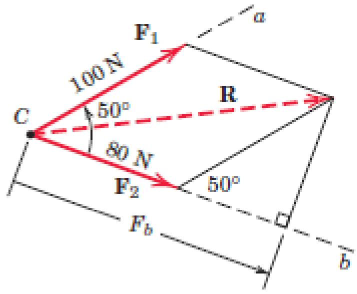
Sample Problem 2/4

Forces **F**1 and **F**2 act on the bracket as shown. Determine the projection *Fb* of their resultant **R** onto the *b*-axis.



Solution. The parallelogram addition of $\mathbf{F}_1$ and $\mathbf{F}_2$ is shown in the figure. Using the law of cosines gives us

$R^2 = (80)^2 + (100)^2 - 2(80)(100)\cos 130^o$ $R = 163.4$ N

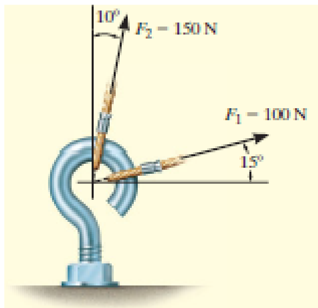


The figure also shows the orthogonal projection $\mathbf{F}_b$ of **R** onto the *b*-axis. Its length is

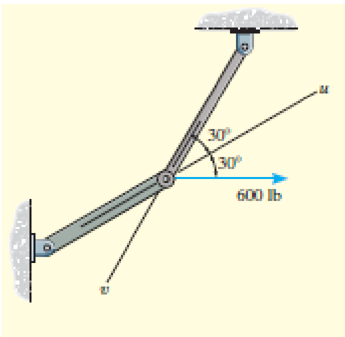
$F_b = 80 + 100\cos 50^o = 144.3$ N *Ans.*

Note that the components of a vector are in general not equal to the projections of the vector onto the same axes. If the *a*-axis had been perpendicular to the *b*-axis, then the projections and components of **R** would have been equal.

(H.W.) Problems:

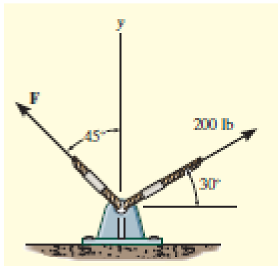


1- The screw eye in Fig. 2–1 a is subjected to two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$ Determine the magnitude and direction of the resultant force.



(a)

2- Resolve the horizontal 600-lb force in Fig. 2–2 a into components acting along the *u* and *v* axes and determine the magnitudes of these components.



(a)

3- Determine the magnitude of the component force **F** in Fig. 2–3 a and the magnitude of the resultant force $\mathbf{F_R}$ if $\mathbf{F_R}$ is directed along the positive y axis.

2/4 MOMENT

This rotational tendency is known as the *moment* **M** of the force. Moment is also referred to as *torque*.

Moment about a Point

Figure 2/8*b* shows a two-dimensional body acted on by a force **F** in its plane. The magnitude of the moment or tendency of the force to rotate the body about the axis *O-O* perpendicular to the plane of the body is proportional both to the magnitude of the force and to the *moment arm d*, which is the perpendicular distance from the axis to the line of action of the force. Therefore, the magnitude of the moment is defined as

$M = F\,d$ **(2/5)**

The basic units of moment in SI units are newton-meters (N.m), and in the U.S. customary system are pound-feet (lb-ft).

Moment directions may be accounted for by using a stated sign convention, such as a plus sign (+) for counterclockwise moments and a minus sign (-) for clockwise moments, or vice versa.

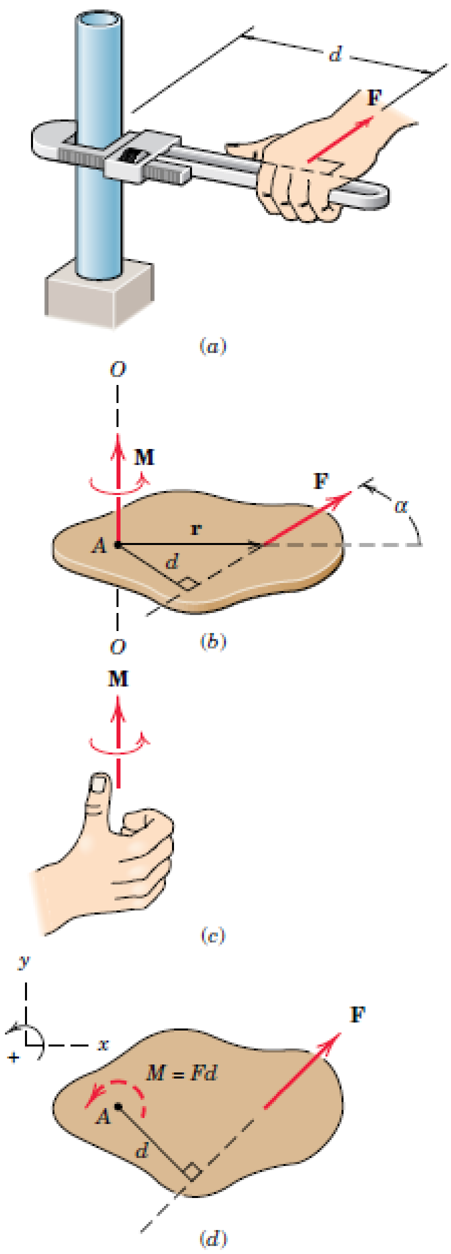


Figure 2/8

The Cross Product

The moment of **F** about point *A* of Fig. 2/8*b* may be represented by the cross-product expression

$\mathbf{M} = \mathbf{r} \times \mathbf{F}$ **(2/6)**

Where **r** is a position vector which runs from the moment reference point *A* to *any* point on the line of action of **F**. The magnitude of this expression is given by

$M = Fr \sin\alpha = Fd$ **(2/7)**

Varignon's Theorem

Varignon's theorem, which states that the moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.

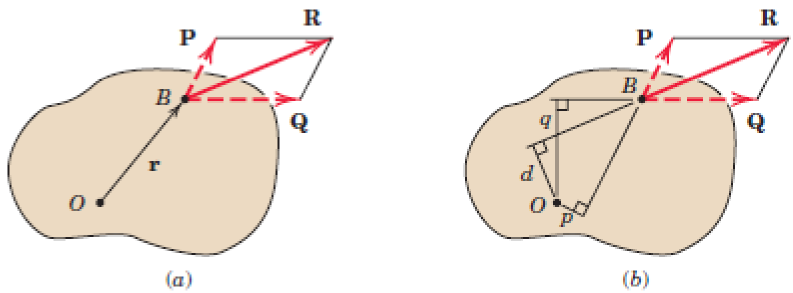


Figure 2/9

$\mathbf{M}_O = \mathbf{r} \times \mathbf{R}$

$\mathbf{R} = \mathbf{Q} + \mathbf{P}$,

$\mathbf{r} \times \mathbf{R} = \mathbf{r} \times (\mathbf{P} + \mathbf{Q})$

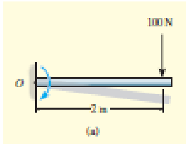
$\mathbf{M}_O = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q}$ **(2/8)**

$M_O = Rd = -pP + qQ$

(H.W.) EXAMPLE 4.1

For each case illustrated in Fig. 4–4, determine the moment of the force about point *O*.

SOLUTION *(SCALAR ANALYSIS)*



The line of action of each force is extended as a dashed line in order to establish the moment arm *d* . Also illustrated is the tendency of rotation of the member as caused by the force. Furthermore, the orbit of the force about *O* is shown as a colored curl. Thus,

Fig. 4–4 *a* M_O = (100 N)(2 m) = 200 N . m *Ans.*

Fig. 4–4 *b* M_O = (50 N)(0.75 m) = 37.5 N . m *Ans.*

Fig. 4–4 *c* M_O = (40 lb)(4 ft + 2 cos 30_ ft) = 229 lb . ft *Ans.*

Fig. 4–4 *d* M_O = (60 lb)(1 sin 45_ ft) = 42.4 lb . ft *Ans.*

Fig. 4–4 *e* M_O = (7 kN)(4 m - 1 m) = 21.0 kN . m *Ans.*

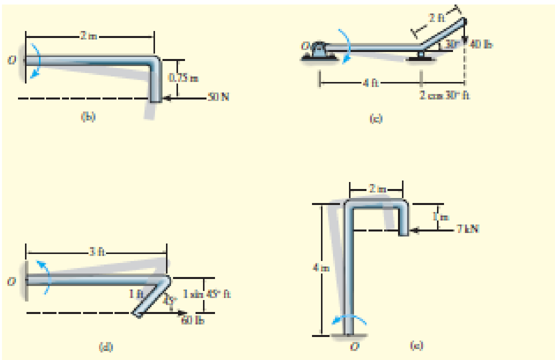


Fig. 4–4

(H.W.) EXAMPLE 4.2

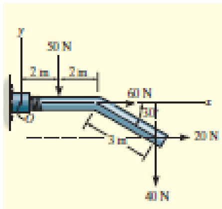


Fig. 4–5

Determine the resultant moment of the four forces acting on the rod shown in Fig. 4–5 about point *O*.

SOLUTION

Assuming that positive moments act in the +**k** direction, i.e., counterclockwise, we have

+ $(M_R)_O$ = _*Fd*;

$(M_R)_O$ = -50 N(2 m) + 60 N(0) + 20 N(3 sin 30° m) -40 N(4 m + 3 cos 30° m)

$(M_R)_O$ = -334 N # m = 334 N. m *Ans.*

For this calculation, note how the moment-arm distances for the 20-N and 40-N forces are established from the extended (dashed) lines of action of each of these forces.

(H.W.) EXAMPLE 4.3

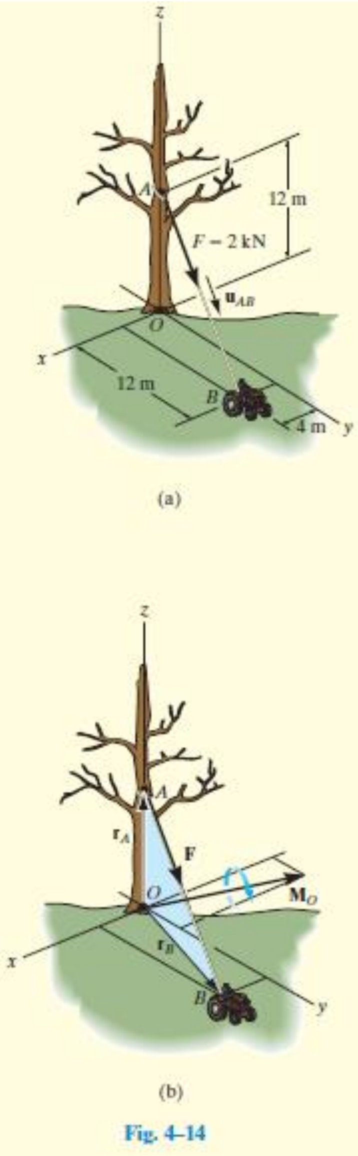


Fig. 4–14

Determine the moment produced by the force **F** in Fig. 4–14 *a* about point *O*. Express the result as a Cartesian vector.

SOLUTION

As shown in Fig. 4–14 *b* , either $\mathbf{r}_A$ or $\mathbf{r}_B$ can be used to determine the moment about point *O* . These position vectors are

$\mathbf{r}_A = \{12\mathbf{k}\}$ m and $\mathbf{r}_B = \{4\mathbf{i} + 12\mathbf{j}\}$ m

Force **F** expressed as a Cartesian vector is

$$\mathbf{F} = F\mathbf{u}_{AB} = 2\text{ kN}\left[\frac{\{4\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}\}\text{ m}}{\sqrt{(4\text{ m})^2 + (12\text{ m})^2 + (-12\text{ m})^2}}\right]$$

$$= \{0.4588\mathbf{i} + 1.376\mathbf{j} - 1.376\mathbf{k}\}\text{ kN}$$

Thus

$$\mathbf{M}_O = \mathbf{r}_A * \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$= [0(-1.376) - 12(1.376)]\mathbf{i} - [0(-1.376) - 12(0.4588)]\mathbf{j} + [0(1.376) - 0(0.4588)]\mathbf{k}$

$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}$ kN.m *Ans.*

Or

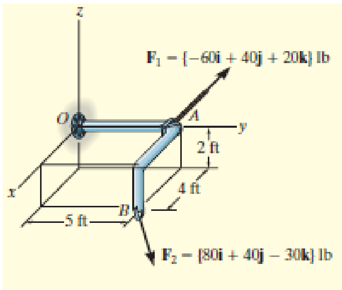
$$\mathbf{M}_O = \mathbf{r}_B * \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 12 & 0 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$= [12(-1.376) - 0(1.376)]\mathbf{i} - [4(-1.376) - 0(0.4588)]\mathbf{j} + [4(1.376) - 12(0.4588)]\mathbf{k}$

$= \{-16.5\mathbf{i} + 5.51\mathbf{j}\}$ kN.m *Ans.*

NOTE: As shown in Fig. 4–14 *b* , $\mathbf{M}_O$ acts perpendicular to the plane that contains **F**, $\mathbf{r}_A$, and $\mathbf{r}_B$. Had this problem been worked using $M_O = Fd$, notice the difficulty that would arise in obtaining the moment arm *d* .

(H.W.) EXAMPLE 4.4

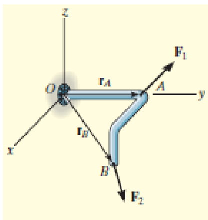


(a)

Two forces act on the rod shown in Fig. 4–15 *a* . Determine the resultant moment they create about the flange at *O* . Express the result as a Cartesian vector.

SOLUTION

Position vectors are directed from point *O* to each force as shown in Fig. 4–15 *b* . These vectors are



(b)

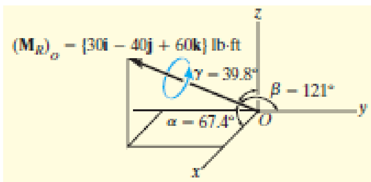
$\mathbf{r}_A = \{5\mathbf{j}\}$ ft , $\mathbf{r}_B = \{4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}\}$ ft

The resultant moment about *O* is therefore

$(\mathbf{M}_R)_O = \sum(\mathbf{r} * \mathbf{F}) = \mathbf{r}_A * \mathbf{F}1 + \mathbf{r}B * \mathbf{F}2$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix}$$

$= [5(20) - 0(40)]\mathbf{i} - [0]\mathbf{j} + [0(40) - (5)(-60)]\mathbf{k} + [5(-30) - (-2)(40)]\mathbf{i} - [4(-30) - (-2)(80)]\mathbf{j} + [4(40) - 5(80)]\mathbf{k} = \{30\mathbf{i} - 40\mathbf{j} + 60\mathbf{k}\}$ lb.ft *Ans.*

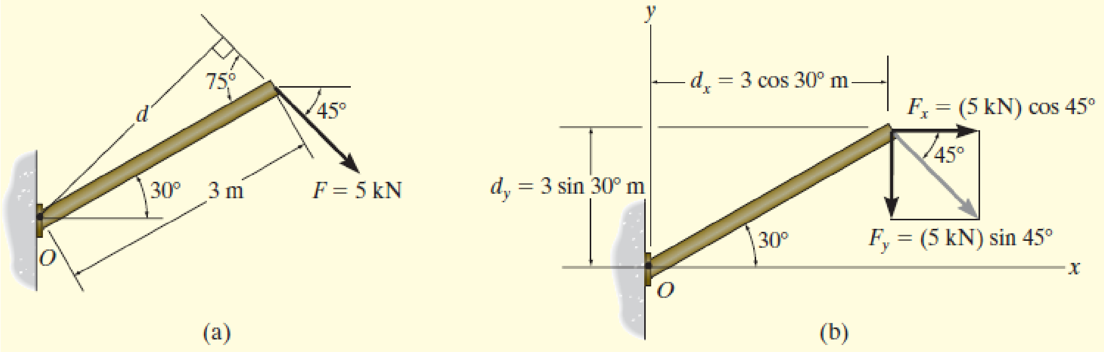


(c)

NOTE: This result is shown in Fig. 4–15 *c* . The coordinate direction angles were determined from the unit vector for $(\mathbf{M}_R)_O$. Realize that the two forces tend to cause the rod to rotate about the moment axis in the manner shown by the curl indicated on the moment vector.

(H.W.) EXAMPLE 4.5

Determine the moment of the force in Fig. 4–18 *a* about point *O* .



SOLUTION I

The moment arm *d* in Fig. 4–18 *a* can be found from trigonometry.

$d = (3\text{ m}) \sin 75^o = 2.898\text{ m}$

Thus,

$M_O = Fd = (5\text{ kN})(2.898\text{ m}) = 14.5\text{ kN.m}$ ↻ *Ans.*

Since the force tends to rotate or orbit clockwise about point *O* , the moment is directed into the page.

SOLUTION II

The *x* and *y* components of the force are indicated in Fig. 4–18 *b* .

Considering counterclockwise moments as positive, and applying the principle of moments, we have

↺ + M_O = -*Fxdy* - *Fydx*

= -(5 cos 45_ kN)(3 sin 30_ m) - (5 sin 45_ kN)(3 cos 30_ m)

= -14.5 kN.m = 14.5 kN.m ↻ *Ans.*

SOLUTION III

The *x* and *y* axes can be set parallel and perpendicular to the rod's axis as shown in Fig. 4–18 *c* . Here **F***x* produces no moment about point *O* since its line of action passes through this point. Therefore,

↺ + M_O = -*Fy dx*

= -(5 sin 75_ kN)(3 m)

= -14.5 kN.m = 14.5 kN.m ↻ *Ans.*

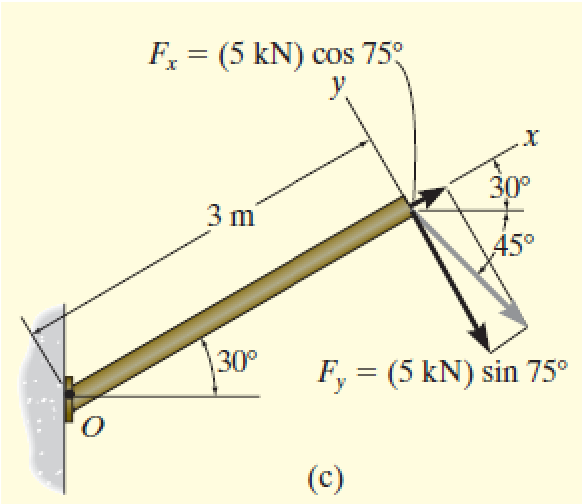


Fig. 4–18

(H.W.) EXAMPLE 4.6

Force **F** acts at the end of the angle bracket in Fig. 4–19 *a*. Determine the moment of the force about point *O*.

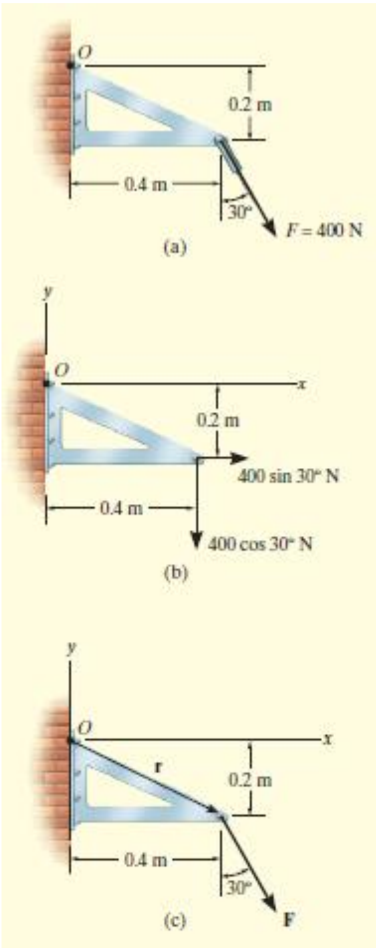


Fig. 4-19

SOLUTION

I *(SCALAR ANALYSIS)*

The force is resolved into its *x* and *y* components, Fig. 4–19 *b* , then

$+M_O = 400 \sin 30^\circ\ N(0.2\ m) - 400 \cos 30^\circ\ N(0.4\ m)$

$= -98.6\ N.m = 98.6\ N.m$

or

$\mathbf{M}_O = \{-98.6\mathbf{k}\}$ N.m *Ans.*

SOLUTION II *(VECTOR ANALYSIS)*

Using a Cartesian vector approach, the force and position vectors,

Fig. 4–19 *c* , are

$\mathbf{r} = \{0.4\mathbf{i} - 0.2\mathbf{j}\}$ m

$\mathbf{F} = \{400 \sin 30^\circ\ \mathbf{i} - 400 \cos 30^\circ\ \mathbf{j}\}$ N

$= \{200.0\mathbf{i} - 346.4\mathbf{j}\}$ N

The moment is therefore

$$\mathbf{M}_O = \mathbf{r} * \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & -0.2 & 0 \\ 200.0 & -346.4 & 0 \end{vmatrix}$$

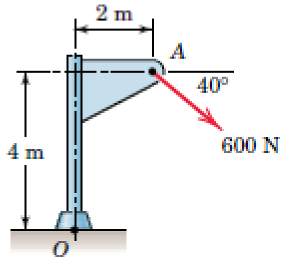
$= 0\mathbf{i} - 0\mathbf{j} + [0.4(-346.4) - (-0.2)(200.0)]\mathbf{k}$

$= \{-98.6\mathbf{k}\}$ N. m *Ans.*

NOTE: It is seen that the scalar analysis (Solution I) provides a more *convenient method* for analysis than Solution II since the direction of the moment and the moment arm for each component force are easy to establish. Hence, this method is generally recommended for solving problems displayed in two dimensions, whereas a Cartesian vector analysis is generally recommended only for solving three dimensional problems.

(C.W.) Sample Problem 2/5

Calculate the magnitude of the moment about the base point *O* of the 600-N force in five different ways.

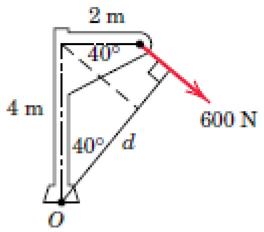


Solution. ***(I)*** The moment arm to the 600-N force is

$$d = 4\cos 40^o + 2\sin 40^o = 4.35 \text{ m}$$

By $M = Fd$ the moment is clockwise and has the magnitude

$$M_o = 600(4.35) = 2610 \text{ N.m} \qquad \textit{Ans.}$$

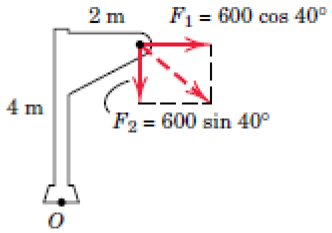


(II) Replace the force by its rectangular components at *A*

$F_1 = 600\cos 40^o = 460$ N, $\quad F_2 = 600\sin 40^o = 386$ N

By Varignon's theorem, the moment becomes

$$M_O = 460(4) + 386(2) = 2610 \text{ N. m} \quad \textit{Ans.}$$



(III) By the principle of transmissibility, move the 600-N force along its line of action to point *B*, which eliminates the moment of the component F_2. The moment arm of F_1 becomes

$$d_1 = 4 + 2\tan 40^o = 5.68 \text{ m}$$

and the moment is

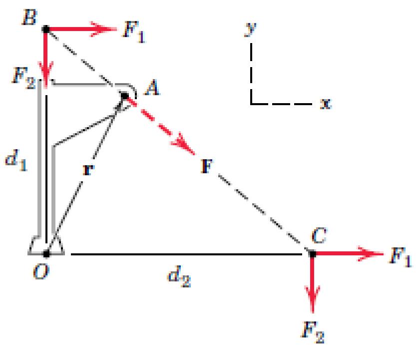
$$M_O = 460(5.68) = 2610 \text{ N. m} \quad \textit{Ans.}$$

(IV) Moving the force to point *C* eliminates the moment of the component F_1. The moment arm of F_2 becomes

$$d_2 = 2 + 4\cot 40^o = 6.77 \text{ m}$$

and the moment is

$$M_O = 386(6.77) = 2610 \text{ N. m} \qquad \textit{Ans.}$$



(V) By the vector expression for a moment, and by using the coordinate system indicated on the figure together with the procedures for evaluating cross products, we have

$$\mathbf{M}_o = \mathbf{r} \times \mathbf{F} = (2\mathbf{i} + 4\mathbf{j}) \times 600(\mathbf{i}\cos 40^o - \mathbf{j}\sin 40^o)$$

$$= -2610\mathbf{K} \text{ N. m}$$

The minus sign indicates that the vector is in the negative *z*-direction. The magnitude of the vector expression is

$$M_O = 2610 \text{ N. m} \qquad \textit{Ans.}$$

Helpful Hints

1- The required geometry here and in similar problems should not cause difficulty if the sketch is carefully drawn.

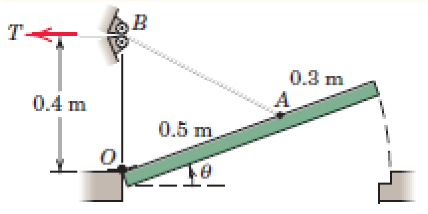
2- This procedure is frequently the shortest approach.

3- The fact that points *B* and *C* are not on the body proper should not cause concern, as the mathematical calculation of the moment of a force does not require that the force be on the body.

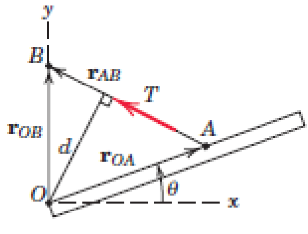
4- Alternative choices for the position vector **r** are $\mathbf{r} = d_1\mathbf{j} = 5.68\mathbf{j}$ m and $\mathbf{r} = d_2\mathbf{i} = 6.77\mathbf{i}$ m.

(C.W.) Sample Problem 2/6

The trap door *OA* is raised by the cable *AB*, which passes over the small frictionless guide pulleys at *B*. The tension everywhere in the cable is *T*, and this tension applied at ***A*** causes a moment $\boldsymbol{M}_O$ about the hinge at *O*. Plot the quantity $\boldsymbol{M}_O/\boldsymbol{T}$ as a function of the door elevation angle θ over the range $0 \le \theta \le 90^o$ and note minimum and maximum values. What is the physical significance of this ratio?



Solution. We begin by constructing a figure which shows the tension force **T** acting directly on the door, which is shown in an arbitrary angular position _. It should be clear that the direction of **T** will vary as θ varies. In order to deal with this variation, we write a unit vector $\mathbf{n}_{AB}$ which "aims" **T**:



$$\boldsymbol{n}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}} = \frac{\mathbf{r}_{OB} - \mathbf{r}_{OA}}{r_{AB}}$$

Using the *x*-*y* coordinates of our figure, we can write

$\mathbf{r}_{OB} = 0.4\mathbf{j}$ m and $\mathbf{r}_{OA} = 0.5(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$ m

So

$\mathbf{r}_{AB} = \mathbf{r}_{OB} - \mathbf{r}_{OA} = 0.4\mathbf{j} - (0.5)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$

$= -0.5\cos\theta\,\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}$ m

and

$$r_{AB} = \sqrt{(0.5\cos\theta)^2 + (0.4 - 0.5\sin\theta)^2}$$

$$= \sqrt{0.41 - 0.4\sin\theta} \quad m$$

The desired unit vector is

$$\boldsymbol{n}_{AB} = \frac{\mathbf{r}_{AB}}{\boldsymbol{r}_{AB}} = \frac{-0.5\cos\theta\,\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}}{\sqrt{0.41 - 0.4\sin\theta}}$$

Our tension vector can now be written as

$$\mathbf{T} = T\mathbf{n}_{AB} = T\left[\frac{-0.5\cos\theta\,\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}}{\sqrt{0.41 - 0.4\sin\theta}}\right]$$

The moment of **T** about point *O*, as a vector, is $\mathbf{M}O = \mathbf{r}_{OB} \times \mathbf{T}$, where $\mathbf{r}_{OB} = 0.4\mathbf{j}$ m, or

$$\mathbf{M}_o = 0.4\,\mathbf{j} \times T\left[\frac{-0.5\cos\theta\,\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}}{\sqrt{0.41 - 0.4\sin\theta}}\right]$$

$$= \frac{0.2\,T\cos\theta\,\mathbf{K}}{\sqrt{0.41 - 0.4\sin\theta}}$$

The magnitude of **M***O* is

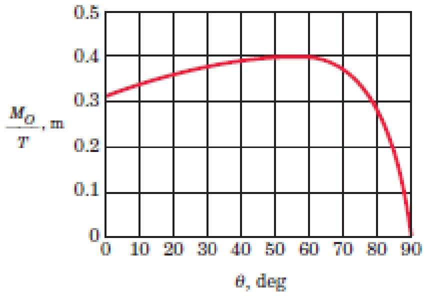
$$M_o = \frac{0.2\,T\cos\theta}{\sqrt{0.41 - 0.4\sin\theta}}$$

and the requested ratio is

$$\frac{\boldsymbol{Mo}}{T} = \frac{0.2\cos\theta}{\sqrt{0.41 - 0.4\sin\theta}} \qquad \textit{Ans.}$$

Helpful Hints

1- Recall that any unit vector can be written as a vector divided by its magnitude. In this case the vector in the numerator is a position vector.



2- Recall that any vector may be written as a magnitude times an "aiming" unit vector.

3- In the expression $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, the position vector **r** runs from the moment center to any point on the line of action of **F**. Here, $\mathbf{r}_{OB}$ is more convenient than $\mathbf{r}_{OA}$.

which is plotted in the accompanying graph. The expression M_O/T is the moment arm *d* (in meters) which runs from *O* to the line of action of **T**. It has a maximum value of 0.4 m at $\theta = 53.1^o$ (at which point **T** is horizontal) and a minimum value of 0 at $\theta = 90^o$ (at which point **T** is vertical). The expression is valid even if *T* varies.

This sample problem treats moments in two-dimensional force systems, and it also points out the advantages of carrying out a solution for an arbitrary position, so that behavior over a range of positions can be examined.

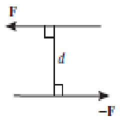


Fig. 4-25

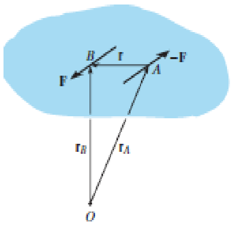


Fig. 4-26

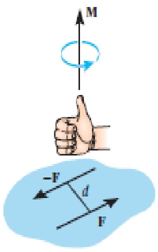


Fig. 4-27

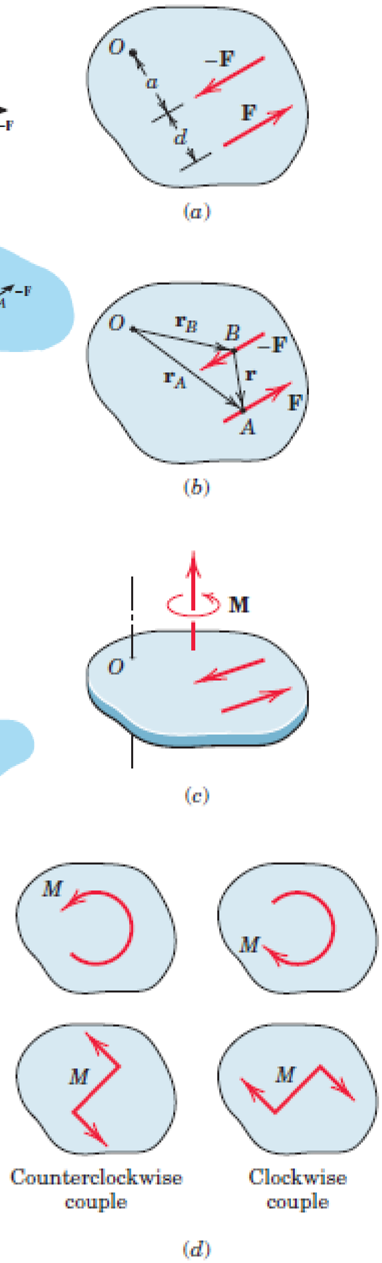


Figure 2/10

2/5 COUPLE

The moment produced by two equal, opposite, and noncollinear forces is called a *couple*.

Consider the action of two equal and opposite forces **F** and **-F** a distance *d* apart, as shown in Fig. 2/10*a*. The combined moment of the two forces about an axis normal to their plane and passing through any point such as *O* in their plane is the couple **M**. This couple has a magnitude

$M = F(a + d) - Fa$

Or

$M = Fd$

Its direction is counterclockwise when viewed from above for the case illustrated.

Note especially that the magnitude of the couple is independent of the distance *a*. the moment of a couple has the same value for all moment centers.

Vector Algebra Method

$\mathbf{M} = \mathbf{r}A \times \mathbf{F} + \mathbf{r}B \times (-\mathbf{F}) = (\mathbf{r}A - \mathbf{r}B) \times \mathbf{F}$

But: $\mathbf{r}A - \mathbf{r}B = \mathbf{r}$,

$\mathbf{M} = \mathbf{r} \times \mathbf{F}$

We may represent **M** by a free vector, as show in Fig. 2/10*c*.

Equivalent Couples

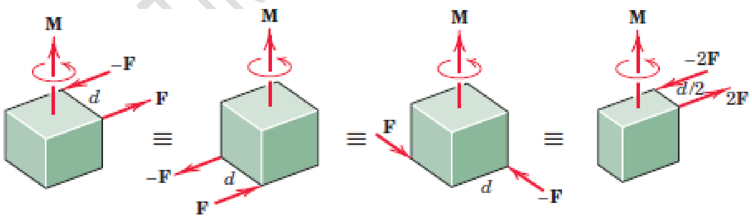


Figure 2/11

Important Points

- A couple moment is produced by two noncollinear forces that are equal in magnitude but opposite in direction. Its effect is to produce pure rotation, or tendency for rotation in a specified direction.
- A couple moment is a free vector, and as a result it causes the same rotational effect on a body regardless of where the couple moment is applied to the body.
- The moment of the two couple forces can be determined about *any point*. For convenience, this point is often chosen on the line of action of one of the forces in order to eliminate the moment of this force about the point.
- In three dimensions the couple moment is often determined using the vector formulation, **M** = **r** * **F**, where **r** is directed from *any point* on the line of action of one of the forces to *any point* on the line of action of the other force **F**.
- A resultant couple moment is simply the vector sum of all the couple moments of the system.

Force–Couple Systems

The effect of a force acting on a body is the tendency to push or pull the body in the direction of the force, and to rotate the body about any fixed axis which does not intersect the line of the force. We can represent this dual effect more easily by replacing the given force by an equal parallel force and a couple to compensate for the change in the moment of the force.

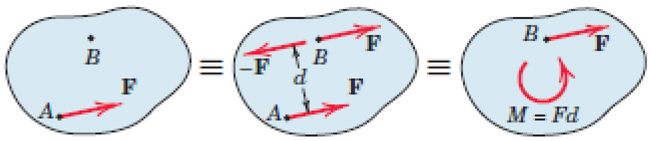


Figure 2/12

The replacement of a force by a force and a couple is illustrated in Fig. 2/12, where the given force **F** acting at point *A* is replaced by an equal force **F** at some point *B* and the counterclockwise couple *M* = *Fd*.

The transfer is seen in the middle figure, where the equal and opposite forces **F** and **-F** are added at point *B* without introducing any net external effects on the body.

Note:

By reversing this process, we can combine a given couple and a force which lies in the plane of the couple (normal to the couple vector) to produce a single, equivalent force. Replacement of a force by an equivalent force–couple system, and the reverse procedure, have many applications in mechanics and should be mastered.

For Example, to Simplification of forces and Couple Moments in x - y plane:

We can replace any number of forces and moment couple to one force in specified point like (E), we can do it by this procedure:

1- Find the $\sum F x$, $\sum F y$ for all forces.

2- Find the resultant force $F = \sqrt{\sum Fx^2 + \sum Fy^2}$

3- Calculate the slope angle with the x- axis as: $\theta = \tan^{-1}\frac{\sum Fy}{\sum Fx}$

4- Calculate the moment $\sum M$ *for all forces* around the specified point (E).

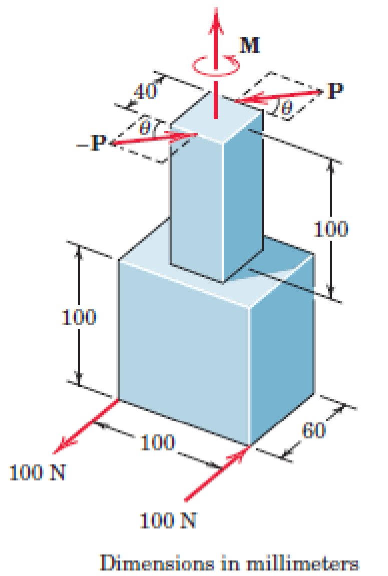
5- Calculate the distance (d) and then use the equation:

$\sum M_E = \sum F_{Ry}\, d$

Or Calculate the component of *Fr* which produce the moment **M** around point E.

Sample Problem 2/7

The rigid structural member is subjected to a couple consisting of the two 100-N forces. Replace this couple by an equivalent couple consisting of the two forces **P** and **-P**, each of which has a magnitude of 400 N. Determine the proper angle θ.



Solution. The original couple is counterclockwise when the plane of the forces is viewed from above, and its magnitude is

[*M* = *Fd*] $M = 100(0.1) = 10$ N.m

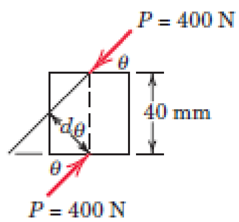
The forces **P** and **-P** produce a counterclockwise couple

$M = 400(0.040)\cos\theta$

1-Equating the two expressions gives

$10 = (400)(0.040)\cos\theta$

$\theta = \cos^{-1}\frac{10}{400*0.04} = \cos^{-1}\frac{10}{16} = 51.3°$ *Ans.*

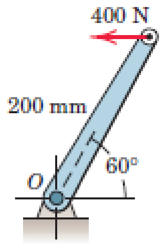


Helpful Hint

1- Since the two equal couples are parallel free vectors, the only dimensions which are relevant are those which give the perpendicular distances between the forces of the couples.

Sample Problem 2/8

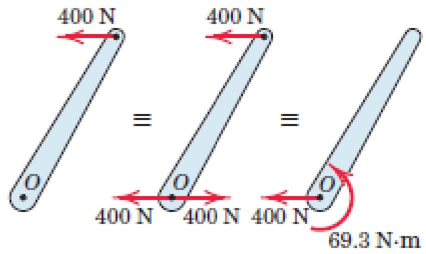
Replace the horizontal 400-N force acting on the lever by an equivalent system consisting of a force at *O* and a couple.



Solution. We apply two equal and opposite 400-N forces at *O* and identify the counterclockwise couple

$[M = Fd]$ $M = 400(0.200 \sin 60^o) = 69.3$ N.m *Ans.*

1-Thus, the original force is equivalent to the 400-N force at *O* and the couple as shown in the third of the three equivalent figures.



Helpful Hint

1- The reverse of this problem is often encountered, namely, the replacement of a force and a couple by a single force. Proceeding in reverse is the same as replacing the couple by two forces, one of which is equal and opposite to the 400-N force at *O*. The moment arm to the second force would be $M/F = 69.3/400 = 0.1732$ m, which is 0.2 sin 60°, thus determining the line of action of the single resultant force of 400 N.

(H.W.)EXAMPLE 4.10

Determine the resultant couple moment of the three couples acting on the plate in Fig. 4–30.

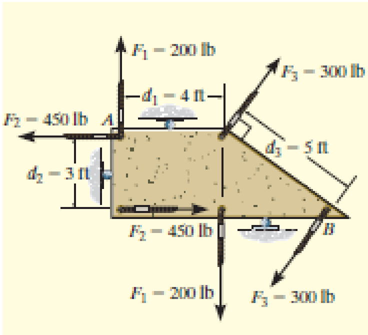


Fig. 4–30

SOLUTION

As shown the perpendicular distances between each pair of couple forces are $d1 = 4$ ft, $d2 = 3$ ft, and $d3 = 5$ ft . Considering counterclockwise couple moments as positive, we have

$+ M_R = \sum M;\ M_R = -F_1 d_1 + F_2 d_2 - F_3 d_3$

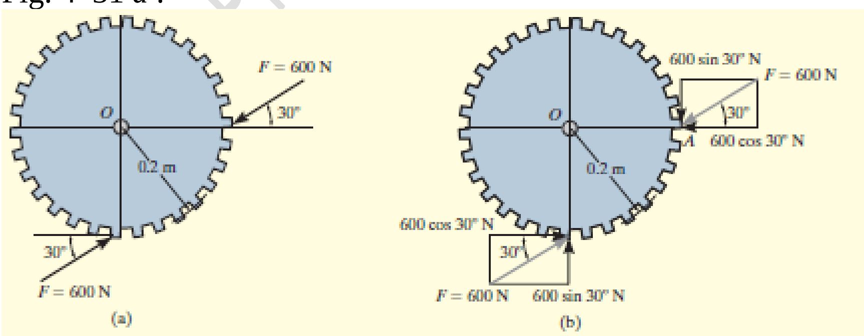
$= -(200 \text{ lb})(4 \text{ ft}) + (450 \text{ lb})(3 \text{ ft}) - (300 \text{ lb})(5 \text{ ft})$

$= -950$ lb. ft $= 950$ lb .ft b *Ans.*

The negative sign indicates that $\mathbf{M}_R$ has a clockwise rotational sense.

(H.W.)EXAMPLE 4.11

Determine the magnitude and direction of the couple moment acting on the gear in Fig. 4–31 *a* .



SOLUTION

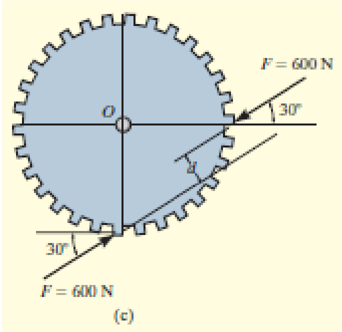


Fig. 4–31

The easiest solution requires resolving each force into its components as shown in Fig. 4–31 *b* . The couple moment can be determined by summing the moments of these force components about any point, for example, the center *O* of the gear or point *A* . If we consider counterclockwise moments as positive, we have

↺ $+M = \sum M_O$; $M = (600 \cos 30^o \text{ N})(0.2 \text{ m}) - (600 \sin 30^o \text{ N})(0.2 \text{ m})$

$= 43.9$ N m ↺ *Ans.*

or

↺ $+M = \sum M_A$; $M = (600 \cos 30^o \text{ N})(0.2 \text{ m}) - (600 \sin 30^o \text{ N})(0.2 \text{ m})$

$= 43.9$ N .m ↺ *Ans.*

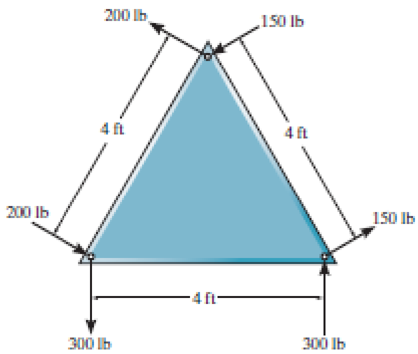
This positive result indicates that **M** has a counterclockwise rotational sense, so it is directed outward, perpendicular to the page.

NOTE: The same result can also be obtained using $M = Fd$, where d is the perpendicular distance between the lines of action of the couple forces, Fig. 4–31 *c* . However, the computation for *d* is more involved.

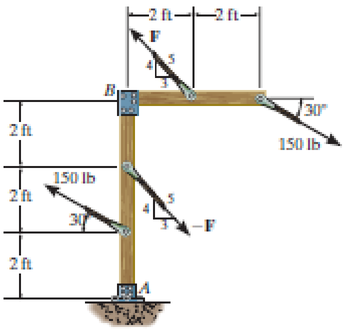
Realize that the couple moment is a free vector and can act at any point on the gear and produce the same turning effect about point *O* .

(H.W.) Problem F4-20

Determine the resultant couple moment acting on the triangular plate.



F4–20

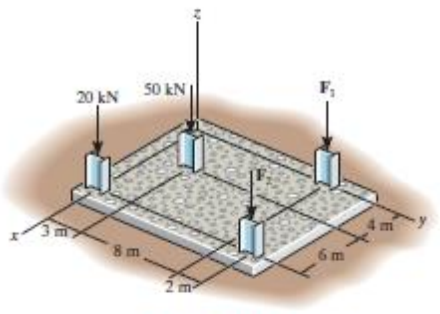


Probs. 4–79/80

(H.W.) 4–80. Determine the required magnitude of force F if the resultant couple moment on the frame is 200 lb .ft , clockwise.

(H.W.) 4–131. The building slab is subjected to four parallel column loadings. Determine the equivalent resultant force and specify its location (x, y) on the slab. Take F1 = 20 kN,

F2 = 50 kN.



Probs. 4–130/131

2/6 RESULTANTS

The *resultant* of a system of forces is the simplest force combination which can replace the original forces without altering the external effect on the rigid body to which the forces are applied.

Equilibrium of a body is the condition in which the resultant of all forces acting on the body is zero. This condition is studied in statics.

The most common type of force system occurs when the forces all act in a single plane, say, the *x-y* plane, as illustrated by the system of three forces **F**1, **F**2, and **F**3 in Fig. 2/13*a*. We obtain the magnitude and direction of the resultant force **R** by forming the *force polygon* shown in part *b*. Thus, for any system of coplanar forces we may write

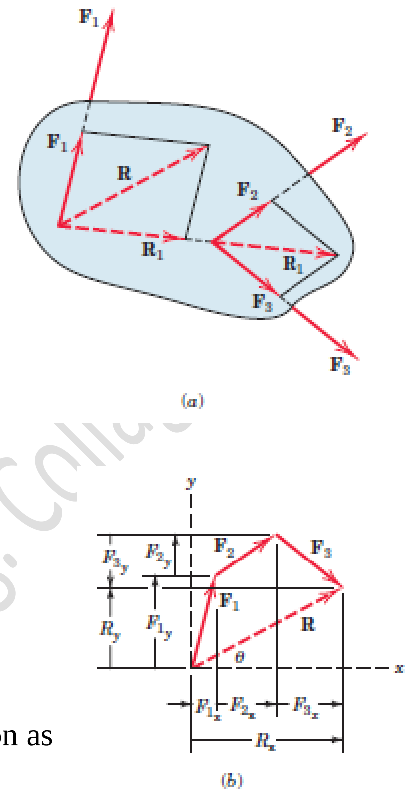


Figure 2/13

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma\mathbf{F}$$
$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$
$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x} \qquad \textbf{(2/9)}$$

Algebraic Method

We can use algebra to obtain the resultant force and its line of action as follows:

1. Choose a convenient reference point and move all forces to that point. This process is depicted for a three-force system in Figs.
2/14*a* and *b*, where *M*1, *M*2, and *M*3 are the couples resulting from the transfer of forces **F**1, **F**2, and **F**3 from their respective original lines of action to lines of action through point *O*.

2. Add all forces at *O* to form the resultant force **R**, and add all couples to form the resultant couple *MO*. We now have the single force– couple system, as shown in Fig. 2/14*c*.

3. In Fig. 2/14*d*, find the line of action of **R** by requiring **R** to have a moment of M_O about point *O*. Note that the force systems of Figs.
2/14*a* and 2/14*d* are equivalent, and that Σ(*Fd*) in Fig. 2/14*a* is equal to *Rd* in Fig. 2/14*d*.

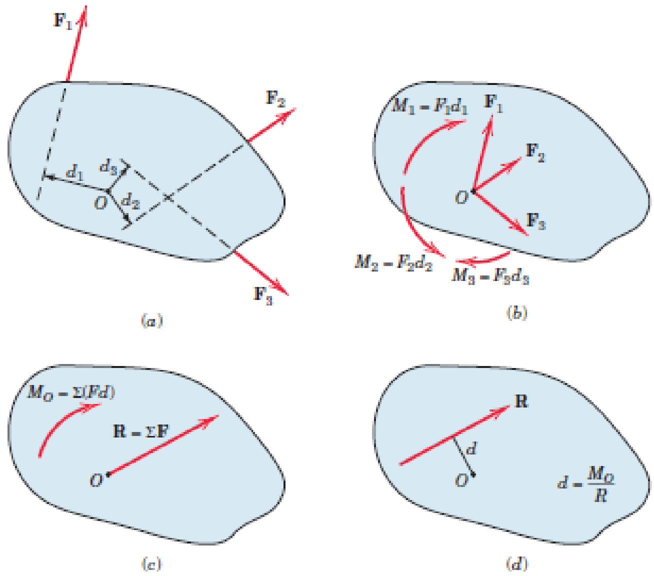


Figure 2/14

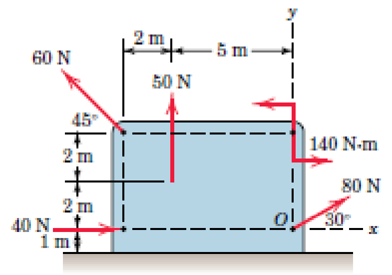
Principle of Moments

This process is summarized in equation form by

$$\mathbf{R} = \Sigma\mathbf{F}$$
$$M_O = \Sigma M = \Sigma(Fd) \qquad \textbf{(2/10)}$$
$$Rd = M_O$$

Sample Problem 2/9

Determine the resultant of the four forces and one couple which act on the plate shown.



Solution. Point O is selected as a convenient reference point for the force–couple system which is to represent the given system.

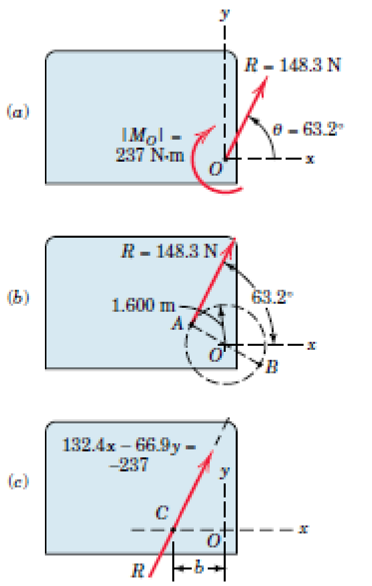
$[R_x = \Sigma F_x]$ $\quad Rx = 40 + 80 \cos 30^o - 60 \cos 45^o = 66.9$ N *Ans.*

$[R_y = \Sigma F_y]$ $\quad Ry = 50 + 80 \sin 30^o + 60 \cos 45^o = 132.4$ N *Ans.*

$[R = \sqrt{(Rx)^2 + (Ry)^2}\,]$ $\quad R = \sqrt{(66.9)^2 + (132.4)^2} = 148.3$ N *Ans.*

$\theta = \tan^{-1}\frac{Ry}{Rx} = \tan^{-1}\frac{132.4}{66.9} = 63.2^o$ *Ans.*

$[M_O = \Sigma(Fd)]$ $\quad M_O = 140 - 50(5) + 60 \cos 45^o (4) - 60 \sin 45^o (7)$
$= -237$ N.m



The force–couple system consisting of **R** and M_O is shown in Fig. *a*. We now determine the final line of action of **R** such that **R** alone represents the original system.

$[Rd = |M_O|]$ $\quad 148.3d = 237$ $\quad d = 1.600$ m

Hence, the resultant **R** may be applied at any point on the line which makes a 63.2 ° angle with the x-axis and is tangent at point A to a circle of 1.600-m radius with center O, as shown in part b of the figure. We apply the equation $Rd = M_O$ in an absolute-value sense (ignoring any sign of M_O) and let the physics of the situation, as depicted in Fig. *a*, dictate the final placement of **R**. Had M_O been counterclockwise, the correct line of action of **R** would have been the tangent at point B.

The resultant **R** may also be located by determining its intercept distance b to point C on the x-axis, Fig. *c*. With Rx and Ry acting through point C, only Ry exerts a moment about O so that

$R_y\, b = |M_O|$ and $b = \frac{237}{132.4} = 1.792$

Alternatively, the y-intercept could have been obtained by noting that the moment about O would be due to Rx only.

A more formal approach in determining the final line of action of **R** is to use the vector expression

$\mathbf{r} = \mathbf{R} \times \mathbf{M}_O$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is a position vector running from point O to any point on the line of action of **R**. Substituting the vector expressions for **r**, **R**, and $\mathbf{M}_O$ and carrying out the cross product result in

$(x\mathbf{i} + y\mathbf{j}) \times (66.9\mathbf{i} + 132.4\mathbf{j}) = -237\mathbf{k}$

$(132.4x - 66.9y)\mathbf{k} = -237\mathbf{k}$

Thus, the desired line of action, Fig. *c*, is given by

$132.4x - 66.9y = -237$

2-By setting $y = 0$, we obtain $x = -1.792$ m, which agrees with our earlier calculation of the distance b.

Helpful Hints

1- We note that the choice of point O as a moment center eliminates any moments due to the two forces which pass through O. Had the clockwise sign convention been adopted, M_O would have been +237 , with the plus sign indicating a sense which agrees with the sign convention.

Either sign convention, of course, leads to the conclusion of a clockwise moment M_O.

2- Note that the vector approach yields sign information automatically, whereas the scalar approach is more physically oriented. You should master both methods.

(H.W.)EXAMPLE 4.1

For each case illustrated in Fig. 4–4 , determine the moment of the force about point *O* .

SOLUTION *(SCALAR ANALYSIS)*

The line of action of each force is extended as a dashed line in order to establish the moment arm *d* . Also illustrated is the tendency of rotation of the member as caused by the force. Furthermore, the orbit of the force about *O* is shown as a colored curl. Thus,

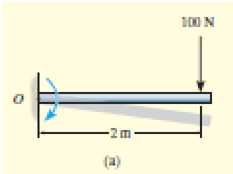


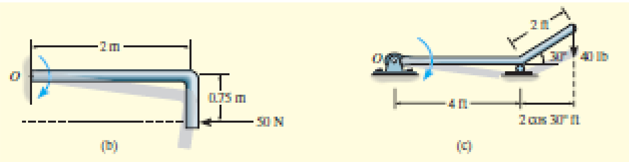
Fig. 4–4 *a* M_O = (100 N)(2 m) = 200 N.m *Ans.*

Fig. 4–4 *b* M_O = (50 N)(0.75 m) = 37.5 N.m *Ans.*

Fig. 4–4 *c* M_O = (40 lb)(4 ft + 2 cos 30_ ft) = 229 lb.ft *Ans.*

Fig. 4–4 *d* M_O = (60 lb)(1 sin 45_ ft) = 42.4 lb.ft *Ans.*

Fig. 4–4 *e* M_O = (7 kN)(4 m - 1 m) = 21.0 kN.m *Ans*



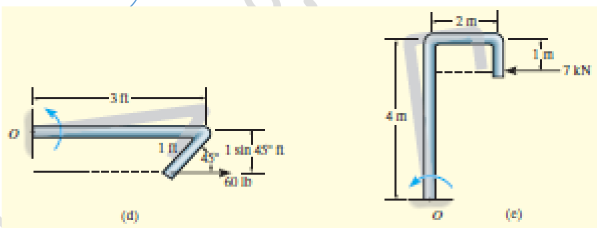


Fig. 4-4

(H.W.) EXAMPLE 4.2

Determine the resultant moment of the four forces acting on the rod shown in Fig. 4–5 about point *O* .

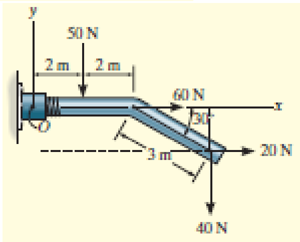


Fig. 4-5

SOLUTION

Assuming that positive moments act in the +**k** direction, i.e., counterclockwise, we have

+ $(M_R)_O$ = _*Fd*;

$(M_R)_O$ = -50 N(2 m) + 60 N(0) + 20 N(3 sin 30° m) -40 N(4 m + 3 cos 30° m)

$(M_R)_O$ = -334 N.m = 334 N.m *Ans.*

For this calculation, note how the moment-arm distances for the 20-N and 40-N forces are established from the extended (dashed) lines of action of each of these forces.